States in Combinatorics

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In this handout, I will introduce the idea of using states to solve computational combinatorics problems, and then demonstrate how to use this powerful method. At the end, there will be practice problems. This is intended for those unfamiliar with states, but I think it will also be beneficial for those somewhat experienced with states to refine your understanding.

1 Introduction by Example

It's pretty hard to just define the method of event states, so instead, I will present an example. If you already know what this method is, you can skip to section 2.

Consider the following theoretical situation:

Example
I am flipping a fair coin with a side denoted as heads and the other denoted as tails. How many times do I have to flip it in order to get heads?

Considering this, we see that there is not actually a singular answer. If we flip it once, there's a 50% chance of us getting heads, but also a 50% chance of us getting tails. So half of the time the answer is one flip? What about the rest of the time? It's not like I can say the answer is $\boxed{\text{One flip half of the time.}}$ That's just absurd!

Instead, we can ask a different question: we can't find a single value for a definite answer, but we can find the **expected value** of number of coin flips I will need in order to get heads. That's just a fancy way of asking *What's the average number of times I will need to flip a coin to get heads for the first time?*

Remark

Let's talk a bit about this idea of **Expected Value.** A lot of state problems involve finding expected value. In layman's terms, the expected value of a random variable is its predicted value. Expected value alone is a topic worth studying, and there are some neat properties of it you may see later in your own studies. In our case though, we consider expected value in its simplest form. In general, the expected value of a variable can be expressed as the weighted average of the possible values it can take.

Take for example, a coin with one side marked with a 4 and the opposite side marked with a 1, and say it is twice as likely to land with the 1 face up. Then the expected value of the face-up quantity is not 1 or 4. It is twice as likely to turn up as a 1, so there's a $\frac{2}{3}$ chance it lands with 1 up, and a $\frac{1}{3}$ chance it lands with 4 up. So then the weighted average is $\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4 = 2$.

Well, let's consider x to be this expected number of flips in order to get flip heads. Our goal is to compute x. Note that we can rephrase the problem is **"How many flips does it take to get from having no heads flipped to one heads flipped?"** Every coin flip will either get us 0 or 1 heads flipped, with equal probability. Let's split this into cases:

- In the first case, we flip a heads. This means we need exactly 1 flip. Nice and easy!
- In the second case, we flip a tails. Darn it! But we notice that flipping the tails got us exactly where we started: 0 heads flipped. Hence, we will need (on average) x more flips in order to get a heads. So then there are (on average) $x + 1$ flips needed (including our initial flip) to get to the state of having one heads flipped if we flip tails first.

Now, the expected value is the average of all cases, so then we have

$$x = \frac{1 + (1 + x)}{2},$$

and its not hard to see that $\boxed{x = 2}$ is our answer.

The solution above is an oversimplified application of states. We solved the problem by considering the two states in this problem: having 0 heads and having 1 head.

However, states of course isn't the only way to solve problems like this one. If you didn't see the other solution, then, well let's just look at it to see *why* the method of considering states is one we should practice and use.

Remark ("Brainless Bash" Solution)
Notice that another way to arrive at the answer is by just writing it out in a less meaningful way:

$$\begin{aligned}
x &= \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \cdots \\
\frac{x}{2} &= \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 + \cdots \\
x - \frac{x}{2} &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\
\frac{x}{2} &= 1
\end{aligned}$$

It turns out that these two solutions are sort of doing the same thing, it's just that the approach is different. Our first approach using states gives us the relation $x = \frac{1+(1+x)}{2}$ via a logical interpretation, and the brainless bash approach tries to explicitly solve for x with an infinite sum, and then finds that we can subtract $\frac{x}{2}$ from both sides to get our infinite geometric sum.

Notice that the states solution requires a bit more thinking, while this one is more brainless bash. In this case, the brainless bash is a fairly easy solution, but other problems will prove difficult to approach without any additional insight. So, for all the problems on this handout, *please* don't use brainless bash to solve any of them, unless it is for fun *after* you use states. States may be confusing at first, but trust me, you will want to know how to solve problems using states. How do you use states? I'm glad you asked...

2 States: the User Manual

First, let's clear some ambiguity.

The Name of the Method

We consider the method used in the previous section as ***"states,"*** but this is not really a very helpful name for those starting out. Consider the following to see what I mean:

- We call the method "states."
- The method involves considering the states of a situation.

We say that we use "states" to solve a problem to really say that we are considering the states of a situation in the problem, and their relation to the other states. (This makes it pretty hard to define the method of states without using the word "states.")

2.1 When do I use States?

Whenever you learn a new technique, method, or theorem, it is crucial that you understand which situations to apply it in. So, when looking at the examples and problems in this document, make sure you don't use states because this handout is about states; rather, it is best if you see *why* one would think about using states based on the context of the problem. I will do my best to aid in this by introducing motivating factors here and also for the example problems.

Here are some indicators that make us consider using a states solution:

1. **Getting from point A to point B.** Frequently, a problem will ask us the probability of / the number of steps needed to get from one place to another, given randomness of movement (i.e, not optimal strategy). This is almost always an indication that states will be useful, because we can relate the states of positions needed to be crossed between point A and B.
2. **If/When _______ happens, _______.** If the second blank isn't terminating the process, then we know there are going to be intermediate states. It will do good to use states in order to see how we relate the condition with the rest of the problem.
3. **Ants randomly crawling and Frogs randomly hopping on lily pads.** I don't know why, but an extremely large number of problems use this flavortext in states problems.

Of course, these reasons aren't the only motivating factors for a states solution, and they definitely guarantee a solution using states. Everything is contextual, but these are some things to look out for. Solving the problems and looking at the examples will help build you intuition for knowing when states will be useful.

2.2 How do I use States?

Of course, practice will make it easier for you to solve other problems with states. However, for the neophytes, consider these useful tips for solving a problem using states:

1. **Draw a State Diagram!** Any problem solved using states has more than one state. In order to get the right answer, we need to make sure we are considering *all* states. One of the best ways to organize is drawing a state diagram; a diagram which indicates the possible states, the relationship between *adjacent* (or directly connected) states, and the probability of going from any state to an adjacent state.

2. **Know how states relate to other states.** We can do this using state diagrams, but we need to first know *what* it means for a state to be related to another state. Let's say that we can get from state A to state B without going through any other states. How do I get from state A to state B? Is there a probability that I go to state B from state A? Does the number of moves I made increase if I go from state A to state B? What changes? This must be considered when solving a problem with states. If this isn't clear, don't worry. This will be more clear once we do some examples.

3. **Write the equations out.** We will be able to extract equations from the state diagram. If you find that you can't get an answer from your equations, or that your answer is wrong, look back at the problem. Did you miss anything? And the state diagram, too. Did I misinterpret an equation? Did I forget something? At first, solving a problem with states may seem unnatural, unintuitive, unfriendly, and un-everything. I can reassure you that these un-'s will go away the more you practice; you will become more comfortable with the finer details quickly.

3 Examples

Don't worry if you don't have the hang of states yet. That's what this section is for! The first subsection here is for those who are learning states for the first time. We will slowly build up to having more complicated problems with more nuances involving states. The second subsection is going to include various examples of states solutions to competition questions. Start wherever you like!

3.1 Basic and Introductory Examples

First, let's do a problem which involves more than 1 case.

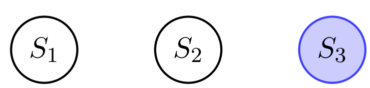
Example 3.1
Kwu is bored. He flips a fair quarter continuously until he gets a heads immediately followed by a tails. What is the expected number of times he will flip the coin?

It's always a good idea to draw a state diagram. After you get a good grip on states, you may not need a state diagram, but for now, let's use one. Since I'm such a great guy, I'll draw one for you!

First, let's consider what our states are. We aren't making our states just like "Heads flipped then tails flipped then heads flipped" or something like that. Rather, the idea here is to consider the progress towards the end goal. So there's a state with 0 progress, in other words, we haven't ever flipped heads yet. Let's call this state S_1. Additionally, there's the state of some progress, in other words, the last flip we made was heads. Let's call this S_2. Finally, let's call the state where we have flipped heads and then tails S_3.

Exercise. Convince yourself that these are all the possible states in this problem.

Okay, now let's draw what we have so far.

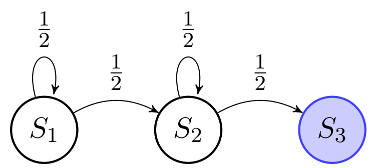


This isn't the completed state diagram. We need to know how the nodes connect to each other! Well, they are related in the following ways:

- First, we know that if we are at S_1, we have flipped zero heads. There's a 50% chance that after a coin flip, we get a tails and make no progress.
- There's also a 50% chance that from S_1, we flip heads then end up at S_2. Hurrah!

- We flip heads at S_2, and then we end back at S_2 with 50% chance.
- Or, we flip tails at S_2, and we end up at S_3.
- From S_3, we have accomplished the goal. Kwu can stop flipping the coin, and the process is terminated. Yay!

Note that from S_2, we never go back to S_1, because flipping tails at S_2 makes us win, and flipping heads keeps us at S_2. This won't always be the case. Anyways, let's add in our connections to the state diagram.



This is our state diagram! What do we do now? This is where stuff gets a bit more confusing, so bear with me. We are going to assign variables E_1, E_2, E_3 to their respective states. We will represent E_1 as the expected number of flips needed to get to S_3, and similarly for E_2 and E_3.

- Suppose we are at S_1. We will need E_1 flips to get to S_3, so we want to compute E_1. Notice that at S_1, we will with 0.5 probability end up back at S_1, and the other half of the time we will go to S_2. If we end up at S_1, we will need E_1 more moves to get to S_3. But if we end up at S_2, we will only need E_2 moves to get to S_3. So we take the average:
$$E_1 = \frac{(E_1 + 1) + (E_2 + 1)}{2}.$$
Note that the $+1$ in both of the fractions comes from the fact that we need to add one coin flip, due to the fact that we flipped a coin.[1]
- Now, we need to find E_2, since our goal is to compute E_1, and E_1 is in terms of E_1 and E_2. Notice that $E_2 = \frac{(E_3+1)+(E_2+1)}{2}$, by similar logic to our first step.
- Finally, E_3 is obviously 0, since we don't need to do anything, as S_3 is our ending step.

[1]If this doesn't make sense: Consider the case in which we flip tails. Then we end up back at S_1, so we will need E_1 more moves to get to S_3. So then we will be at $E_1 + 1$ moves needed to get to S_3. Then, since we take the average of the possibilities, we have the over two.

If you didn't notice, we can actually solve for E_1 now, so the rest is straightforward algebra! Solving for E_1 yields $E_1 = 4$. And we are done! Isn't that great?

Let's do the same problem but a bit more complicated. You will see that there is a fairly large difference.

Example 3.2
Kwu is bored. He flips a fair quarter continuously until he gets heads immediately followed by tails immediately followed by heads. What's the expected number of times he will flip the coin?

This is very similar to our other problem. Instead, we will have four states though. I will give a state diagram, but I encourage you to draw your own before looking at mine on the following page. If you don't remember how to draw a state diagram, then refer to the previous example problem.

Exercise. Draw a state diagram before reading on.

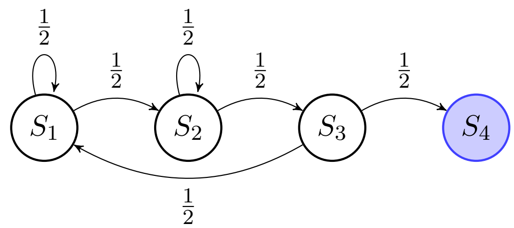
Well, our states here will be

- No progress. In other words, our last two flips were tails followed by tails, if we have had two flips already. If not, then we have either just started or just flipped tails.
- Heads, but not after heads immediately followed by tails.
- Tails after heads.
- Heads, tails, heads. Our goal.

All our connections to other states have probability 50%, since its a fair coin. So we have

- S_1 can either go to S_1 again or S_2.
- S_2 can either go to S_3 or S_2 again.
- S_3 can either go to S_1 or S_4. It can't go back to itself, since if we flip HTT, then the streak is broken and we have no progress.

If you have drawn the state diagram, then go to the next page and see if your diagram is identical to mine. If not, see why mine is correct and why I set it up the way I did.



Just like in the previous example, we can now write our equations.

- S_4 is the end case, so $E_4 = 0$.
- S_3 can either go to S_4 or S_1. If it goes to S_1, then we will need $E_1 + 1$ steps. However, if we end up at S_4, we will need $E_4 + 1$ steps. And since each happen with equal probability, we take the average. So then $E_3 = \frac{(E_1+1)+(E_4+1)}{2}$.
- S_2 can go to itself or S_3, so $E_2 = \frac{(E_2+1)+(E_3+1)}{2}$.
- S_1 can go to itself or S_2, so $E_1 = \frac{(E_1+1)+(E_2+1)}{2}$.

Make sure you see where I got these equations from. You should be able to solve for E_1 now. Did you get 10? If so, good job! If not, check your work. Don't worry, computational mistakes happen to the best of us.

Now, we have seen how to use states to solve these two basic extensions of our introduction example. Let's see if you can use them for some real competition problems.

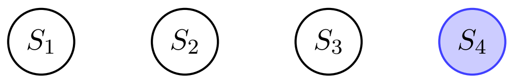
3.2 Examples from Competitions

I recommend that you try all of these problems on your own before seeing how I solve them. This will help you to see where you lack understanding.

Remark (On Motivation)
All the problems here are from actual math competitions. Usually, in a math competition, the problem doesn't say "Use states on this problem." You instead have to think about using states. The factor that inspires us to solve a problem the way we do is called **motivation.** For example, I could be motivated to use constructive counting if a problem asked all the ways I can roll two die and get a sum of eleven, because it's easy to count the number of ways to get a sum of eleven. Try to find the motivating factors for a states solution in the following problems!

Example 3.3 (Classical)
Suppose there are two cans, one is red and the other is blue. We start with three marbles in the red can. We play a game where we choose one marble at random out of the three, then move it to the opposite can. What's the expected number of moves needed for all the marbles to be in the blue can?

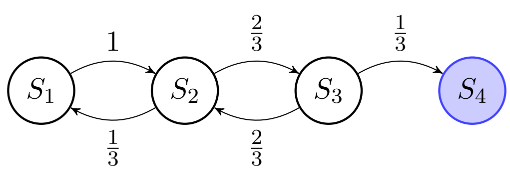
One again, we start by drawing a state diagram. Our states will be something like 3 red 0 blue, 2 red 1 blue, etc. We label these as S_1, S_2, S_3, and S_4.



Now, our connections are going to be fairly different from the last examples, and this is because our previous examples involved 50-50 chances. However, now we don't have identical probabilities every time. Let's see why:

- There's a 100% chance we go from S_1 to S_2. Any marble we choose will be in the red can.
- From S_2, there's a $\frac{2}{3}$ chance we get to S_3, and a $\frac{1}{3}$ chance we go back to S_1.
- From S_3, there's a $\frac{1}{3}$ chance we go to S_4, and a $\frac{2}{3}$ chance we go back to S_2.
- S_4 is our endcase. We finish the process if we end up here.

Coolio! Let's draw our new state diagram with these additions.



Like before, we define E_1, E_2, E_3, and E_4 as the expected number of moves to get from their respective state to S_4. It follows that $E_4 = 0$ and that our answer is E_1. Let's write some equations out:

- $E_4 = 0$, as in all previous examples.
- $E_3 = \frac{(1+E_4)}{3} + \frac{2(E_2+1)}{3}$, which we can derive in a similar fashion to the previous examples.

- $E_2 = \frac{2(1+E_3)}{3} + \frac{1+E_1}{3}$.
- $E_1 = E_2 + 1$.

From here, we just solve like always and get $E_1 = 10$. This is our answer! Thus, we can expect that it will take 10 moves for us to end up with all the marbles in the blue can. If you want, you can try to brainless bash this problem out...It will be very annoying, I can assure you that. This is because we have an unlimited number of possible sequences of moves, and they aren't easily compared. Let's look at another example!

Example 3.4 (2014 AMC10)
In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N-1$ with probability $\frac{N}{10}$ and to pad $N+1$ with probability $1-\frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape without being eaten by the snake?

(A) $\frac{32}{79}$ **(B)** $\frac{161}{384}$ **(C)** $\frac{63}{146}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

Here, we don't have expected value requested. So why would we use states? Well, it turns out states is not solely for computing expected value, even though I have only demonstrated its use with it. In this example, we will use states to calculate probability.

Before we do this, let's consider the motivation behind a states solution. Well, one hint here is that we are given lily pads, and we know that the frog jumps to one of its neighboring lily pads. This is sort of similar to what we did earlier; we always considered going from one state to a different state. In addition, notice that we have an end scenario. The frog will either be eaten or it will escape, and these are both end cases.

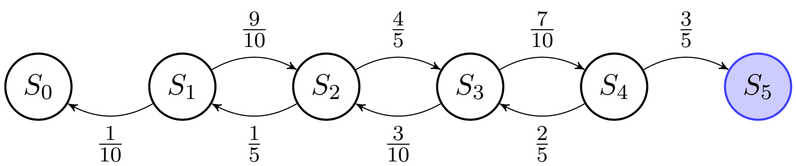
At a first glance, this looks extremely tedious to do with states! We will have eleven lily pads, thus we will need 11 states. Yikes. States will probably work, but also this problem is from the AMC10, meaning that we want our answer and we want it fast. Time to use brain!

Notice that at lily pad 5, we have 50-50 chances of going in either direction. Further more, we can note that if we are at lily pad 9, the probability of escaping is identical to the probability of getting eaten at lily pad 1. Now, it's not hard to see that if I am lily pad 5, the probability of me getting devoured is identical to the probability of me living another day. Thus, by simple logic and symmetry, we can say that the probability of escaping at lily pad 5 is $\frac{1}{2}$. This makes our life much easier!

We can now start by drawing our state diagram. It's clear that our states comprise of being on each lilypad, but in the case of this problem, we have some things we should first look at.

- We have an S_0, which represents the case that we land on lily pad 0 and die. We don't start at this lilypad, despite it being the leftmost in our case. We instead theoretically start from S_1.
- We don't have an S_{10}, because as I said earlier, we know that once we get to S_5, the chances of living are 50-50.

Our connections are pretty obvious from the problem statement. That's a relief.



I will denote P_1 as the probability that from S_1 I end up surviving, and likewise. We have already determined that $P_5 = \frac{1}{2}$ from our earlier logic, which is helpful! Besides that, we have the following relations:

- $P_4 = \frac{3}{5} \cdot P_5 + \frac{2}{5} \cdot P_3$. This is just constructive probability.
- $P_3 = \frac{7}{10} \cdot P_4 + \frac{3}{10} \cdot P_2$
- $P_2 = \frac{4}{5} \cdot P_3 + \frac{1}{5} \cdot P_1$
- $P_1 = \frac{9}{10} \cdot P_2 + \frac{1}{10} \cdot P_0$
- $P_0 = 0$

Then, we can solve for $P_1 = \frac{63}{146}$. AMC10 Problem number 25: it's that easy with states.

Remark

The idea we use here is **symmetry**. Symmetry is powerful, because in states problems, it can give us values of certain states without computation! Symmetry mainly comes up in probability-based questions. Always see if you can make your life easier by reducing the amount of computation!

Let's look at one more problem together, then the rest will be on your own.

Example 3.5 (2016 AIME I)
Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

Alright, this is a states handout so duh, we are going to use states. Disregarding that though, there are many good motivating factors for a states solution. First of all, we know that Freddy hops to an adjacent lattice point **at random,** and we know how to handle random with states. In addition, we are being asked for an expected value, which is also indicative of a states solution. And finally, we also know that we have a frog. I'm joking, but seriously, a nontrivial amount of problems from the AMCs that use frogs are states problems.

So nothing is said about the relevance of the x-coordinate, so really, it doesn't matter. The only thing we care about is the current y-coordinate. Which means, we only have 25 states. Wait. 25 states!? That's a lot! I am *not* spending my time to make a state diagram with 25 states. That's too much work. And you should feel the same way! Even if you do get a substantial amount of time to do the AIME, drawing that many states in a state diagram is unreasonable (I'm pretty sure that 25 states won't fit nicely on a single sheet of paper). However, that should not deter you.

Remark (On State Diagrams)
State diagrams are very helpful tools. They let you visually see what goes on in a problem, and how states are connected to each other, which is why I drew one for every single example up till now. But, they aren't actually crucial to solving states problems. If it seems like you have a reasonable amount of states to draw, go for it!

We would normally find every state and find every connection between the states. But, we aren't going to do that this time. We will instead make one observation at a time.

Let $E(t)$ denote the expected number of moves it takes to get from any point with y-coordinate t to get to a point with y-coordinate 24. We want to compute $E(21)$, and we know that $E(24) = 0$. This is a good start... but not really that helpful. We need something more powerful to get us off our feet.

For all integer k between 1 and 23 inclusive, we know that

$$E(k) = \frac{1}{4} \cdot (E(k-1) + 1) + \frac{1}{4} \cdot (E(k+1) + 1) + \frac{1}{2}(E(k) + 1).$$

So we have an expression for $E(n)$ for n between 1 and 23 inclusive, plus a value for $n = 24$. We're just missing $E(0)$, then. Sure enough, we can write

$$E(0) = \frac{2}{3} \cdot (E(0) + 1) + \frac{1}{3} \cdot (E(1) + 1).$$

Actually, we can rewrite this to be

$$E(0) = E(1) + 3.$$

Now, we have all the information we need. Our next step should be checking the value of $E(1)$. We can substitute into the expression for $E(1)$ to get $E(1) = E(2) + 7$. And after that, we get $E(2) = E(3) + 11$. We found a pattern! This relation can be written as

$$E(n) = E(n+1) + 4(n+1) - 1.$$

This means we can find our answer fairly quickly. I'll leave the rest as an

Exercise (to the reader)**.** Finish the problem from here!

4 Practice Problems

By this point, you should have all the necessary knowledge and tools to tackle these problems. You're on your own! But if you feel like you still need help and can't figure it out, don't worry. Some of these problems are very difficult. There are many great places you can find solutions to these problems, or get help and hints. If you want to contact me specifically, you can also join CNCM, the best Competitive math Discord server. I often hang around there, but you can also ask other brilliant mathematicians. Happy solving!

Problem 1 (2013 SMT)**.** Suppose two equally strong tennis players play against each other until one player wins three games in a row. The results of each game are independent, and each player will win with probability $\frac{1}{2}$. What is the expected value of the number of games they will play?

Problem 2 (WWPMT)**.** Rachel is standing at the middle of a 3 by 3 square grid. Every second, she randomly chooses to walk up, down, left, or right by 1 unit. Rachel stops if she walks outside the square. Find the the expected value of (in other words, on average) the number of steps she makes before she stops. (For example, if she goes left two times, she stops, and the number of steps made is 2)

Problem 3 (2020 AMC10A)**.** A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0), (0, 4), (4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

Problem 4 (2003 HMMT)**.** Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

Problem 5 (2019 AMC10A)**.** Raashan, Sylvia, and Ted play the following game. Each starts with \$1. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives \$1 to that player. What is the probability that after the bell has rung 2019 times, each player will have \$1? (For example, Raashan and Ted may each decide to give \$1 to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have \$0, Sylvia would have \$2, and Ted would have \$1, and and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their \$1 to, and and the holdings will be the same as the end of the second round.)

Problem 6 (DMM 2008)**.** Two cows play a game where each has one playing piece, they begin by having the two pieces on opposite vertices of an octahedron, and the two cows take turns moving their piece to an adjacent vertex. The winner is the first player who moves its piece to the vertex occupied by its opponent's piece. Because cows are not the most intelligent of creatures, they move their pieces randomly. What is the probability that the first cow to move eventually wins?

Problem 7 (2017 AMC12A)**.** A square is drawn in the Cartesian coordinate plane with vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$. A particle starts at $(0, 0)$. Every second it moves with equal probability to one of the eight lattice points (points with integer coordinates) closest to its current position, independently of its previous moves. In other words, the probability is $\frac{1}{8}$ that the particle will move from (x, y) to each of $(x, y + 1)$, $(x + 1, y + 1)$, $(x + 1, y)$, $(x + 1, y - 1)$, $(x, y - 1)$, $(x - 1, y - 1)$, $(x - 1, y)$, $(x - 1, y + 1)$. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or one of the 12 lattice points in the interior of one of the sides of the square. The probability that it will hit at a corner rather than at an interior point of a side is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Problem 8 (MBMT Leibniz)**.** Steven starts with the number 1. Then, he repeats the following procedure N times: if he has the number n, he adds a random integer from 1 to $\gcd(n, 4)$, inclusive, to n. If $N = 2019^{2019^{2019}}$, find the closest integer to $100p$, where p is the probability that Steven's final number is divisible by 4.

Problem 9 (USAMTS 3/4/20)**.** A particle is currently at the point $(0, 3.5)$ on the plane and is moving towards the origin. When the particle hits a lattice point (a point with integer coordinates), it turns with equal probability $45°$ to the left or to the right from its current course. Find the probability that the particle reaches the x-axis before hitting the line $y = 6$.